Chapter 12
Quantum Theory of the Laser

Abstract The quantum theory of the laser was developed in the 1960s principally by the schools associated with H. Haken, W.E. Lamb and M. Lax, see [1, 2, 3, 4]. Haken and Lax independently developed sophisticated techniques to convert operator master equations into c-number Fokker–Planck equations or equivalent Langevin equations.

In this chapter we shall follow the approach of *Scully* and *Lamb* [3] to compute photon statistics and the linewidth of the laser. In the *Scully–Lamb* treatment the pumping is modelled by the injection of a sequence of inverted atoms into the laser cavity. In a usual laser, with a thermal pumping mechanism, a Poisson distributed sequence of inverted atoms is assumed. Introduction of a Bernoulli distribution enables a more general class of pumping mechanisms to be considered, including the case of the regularly pumped laser. Diode lasers with more regular pumping than usual lasers have recently been shown to give rise to sub-shot-noise photocurrent fluctuations.

12.1 Master Equation

A single mode cavity field is excited by a sequence of atoms injected into the cavity. Let t_i be the arrival time of the atom i in the cavity and τ the time spent by each atom in the cavity. The change in the density operator for the field due to the interaction with the ith atom may be represented by

$$\rho(t_i+\tau) = \mathscr{P}(\tau)\rho(t_i)\,. \tag{12.1}$$

The explicit form of $\mathscr{P}(\tau)$ depends on the particular atomic system used in the excitation process. The model we will employ is indicated in Fig. 12.1.

Of the four levels, only levels $|1\rangle$ and $|2\rangle$ are coupled to the intracavity field, which thus are referred to as the lasing levels. Each of these levels may then decay. Level $|1\rangle$ decays to level $|3\rangle$ at a rate γ_1 while level $|2\rangle$ decays to level $|4\rangle$ at a rate γ_2.

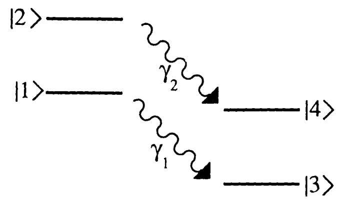

Fig. 12.1 Schematic representation of the four-level atomic model of a laser. Only levels 1 and 2 at coupled to the laser field

We will assume that these decay rates are very much greater than the spontaneous decay rate of level $|2\rangle$ to level $|1\rangle$, and thus we neglect spontaneous emission in the lasing levels. Each atom is prepared in the excited state $|2\rangle$ prior to interaction with the cavity field. In the usual laser system the lifetimes γ_1^{-1} and γ_2^{-1} are much shorter than the time τ spent by each atom in the cavity. This means that each atom rapidly attains a steady state in passing through the cavity and the pump operation $\mathscr{P}(\tau)$ is effectively independent of the time τ. The effect of a single atom on the state of the field may then be written as

$$\rho' = \mathscr{P}\rho \,, \tag{12.2}$$

where we have dropped the time dependence in ρ for simplicity, the prime serving to indicate the state of the field after the passage of a single atom through the cavity. We may represent the initial state of the field quite generally as

$$\rho = \sum_{n,m=0}^{\infty} \rho_{n,m}(0)|n\rangle\langle m| \,. \tag{12.3}$$

In Appendix [12.A] we solve the master equation for the system over the time τ under the assumptions discussed above. The result is

$$\rho' = \sum_{n,m=0}^{\infty} \rho_{n,m}(0)(A_{nm}|n\rangle\langle m| + B_{nm}|n+1\rangle\langle m+1|) \,, \tag{12.4}$$

where the explicit expressions for A_{nm}, B_{nm} are given in the appendix.

We now assume that each atom contributes independently to the field. (This assumption remains valid even if there is more than one atom in the cavity at any time, provided that they are sufficiently dilute.) Thus, if k atoms are passed through the cavity from time 0 to time t the field density operator at time t is given by

$$\rho(t) = \mathscr{P}^k\rho(0) \,. \tag{12.5}$$

More generally, however, not all atoms entering the cavity are prepared in the excited state. Let the probability for an excited atom to enter the cavity between t and $t+\Delta t$ be $r\Delta t$, r being the average injection rate. This defines a Poisson excitation process. Thus the field at time $t+\Delta t$ is made up of a mixture of states corresponding to atomic excitation and no atomic excitation, thus

$$\rho(t+\Delta t) = r\Delta t\mathscr{P}\rho(t) + (1-r\Delta t)\rho(t) \,. \tag{12.6}$$

In the limit $\Delta t \to 0$ we have

$$\frac{\mathrm{d}\rho(t)}{\mathrm{d}t} = r\mathscr{U}\rho(t) \tag{12.7}$$

where

$$\mathscr{U} = \mathscr{P} - 1 \,. \tag{12.8}$$

We must now include the decay of the cavity field through the end mirrors. This is modelled in the usual way by coupling the field to a zero temperature heat bath. Thus the total master equation for the field density operator is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = r\mathscr{U}\rho + \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \,, \tag{12.9}$$

where κ is the cavity decay rate. This is the usual *Scully–Lamb* laser master equation.

In the special case that $\gamma_1 = \gamma_2 = \gamma$ the matrix elements of $\mathscr{U}$ in the number basis are greatly simplified. In this case the master equation in the number basis may be written as

$$\begin{aligned}\frac{\mathrm{d}\rho_{nm}}{\mathrm{d}t} = G\Bigg(&\frac{\sqrt{nm}}{1+(n+m)/2n_\mathrm{s}}\rho_{n-1,m-1}\\ &-\frac{(m+n+2)/2+(m-n)^2/8n_\mathrm{s}}{1+(n+m+2)/2n_\mathrm{s}}\rho_{nm}\Bigg)\\ &+\frac{\kappa}{2}[2\sqrt{(n+1)(m+1)}\rho_{n+1,m+1} - (n+m)\rho_{nm}] \,,\end{aligned} \tag{12.10}$$

where

$$G = \frac{r}{2n_\mathrm{s}} \tag{12.11}$$

and

$$n_\mathrm{s} = \frac{\gamma^2}{4g^2} \,. \tag{12.12}$$

where g is the coupling strength between the cavity and the levels 1 and 2.

we have neglected terms $\propto n_\mathrm{s}^{-2}$ in the denominators of the first two coefficients.

12.2 Photon Statistics

The photon number distribution obeys the equation

$$\begin{aligned}\frac{\mathrm{d}p_n}{\mathrm{d}t} = -G&\left(\frac{n+1}{1+(n+1)/n_\mathrm{s}}p_n - \frac{n}{1+(n/n_\mathrm{s})}p_{n-1}\right)\\ &+\kappa(n+1)p_{n+1} - \kappa n p_n \,.\end{aligned} \tag{12.13}$$

The gain coefficient G is defined by

$$G = \frac{r\gamma_1}{2\gamma + n_s} , \tag{12.14}$$

where $\gamma_+ = (\gamma_1 + \gamma_2)/2$.

If we expand the denominators in (12.13) to first-order an approximate equation for the mean photon number may be obtained, namely

$$\frac{\mathrm{d}\bar{n}}{\mathrm{d}t} = (G-\kappa)\bar{n} - \frac{G}{n_s}(\overline{n^2} + 2\bar{n} + 1) + G . \tag{12.15}$$

If $G > \kappa$ there will be an initial exponential increase in the mean photon number. Thus $G = \kappa$ is the threshold condition for the laser.

The steady state photon number distribution may be deduced directly from (12.13), using the condition of detailed balance. It may be written in the form

$$p_{\mathrm{n}}^{\mathrm{ss}} = \mathscr{N} \frac{(Gn_s/\kappa)^{n+n_s}}{(n+n_s)!} , \tag{12.16}$$

where $\mathscr{N}$ is a normalisation constant. Below threshold $(G < \kappa)$ this distribution may be approximated by a chaotic (thermal) distribution with the mean $\bar{n} = G/(\kappa - G)$ (Exercise 12.1). Above threshold $(G > \kappa)$ the mean and variance are given, to a good approximation, by (Exercise 12.2),

$$\bar{n} = n_s \left(\frac{G}{\kappa} - 1 \right) , \tag{12.17}$$

$$V(n) = \bar{n} + n_s . \tag{12.18}$$

Well above threshold $\bar{n} \gg n_s$ and thus $V(n) \approx \bar{n}$, indicating an approach to Poisson statistics. In Fig. 12.2 we show the exact photon number distribution for below and above threshold. The transition from power law to the Poisson distribution is quite evident.

Photon counting experiments by *Arecchi* [5], *Johnson* et al. [6], and *Morgan* and *Mandel* [7], demonstrated that the photon statistics of a laser well above threshold, approaches a Poisson distribution. In Fig. 12.3 we present the results of photon counting measurements by *Arecchi* on both thermal and laser light. A comparison of the experimental data with the thermal and Poisson distributions is also shown.

12.2.1 Spectrum of Intensity Fluctuations

Equations (12.17 and 12.18) give the photon number fluctuations for the internal cavity mode. This quantity, however, is not directly observable. We must now determine how the photon number fluctuations inside the cavity determine the intensity

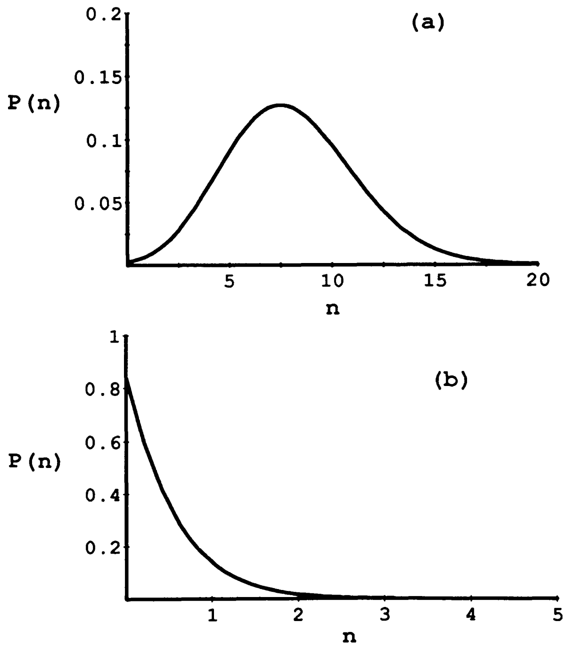

Fig. 12.2 The steady state photon number distribution of a laser operating above and below threshold. In (**a**) $G/\kappa = 5.0$, in (**b**) $G/\kappa = 0.25$. In both cases $n_s = 2$

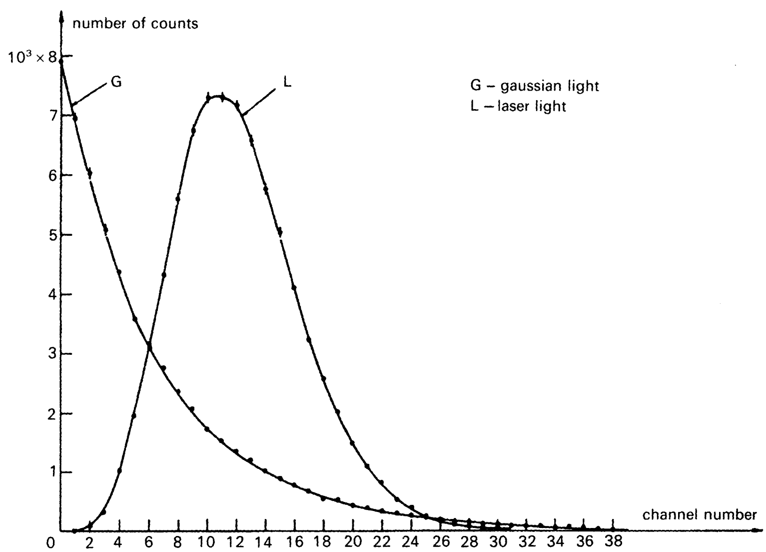

Fig. 12.3 Experimental results for the steady state photon number distribution for a thermal (i.e. Gaussian) light source and a laser operator above threshold. The laser exhibits Poissonian photon number statistics [5]

fluctuations in the many mode field to which it is coupled through the output mirrors. This is an application of the general input/output theory described in Chap. 7.

A single photoelectron detector placed at the output of the cavity measures a photocurrent given by

$$\overline{i(t)} = e\xi \langle b_{\text{out}}^{\dagger}(t) b_{\text{out}}(t) \rangle \,, \tag{12.19}$$

where e is the electronic charge, and $\xi = 2\varepsilon_0 cA/\hbar\omega$ with A the area of the detector surface. (We will assume unit quantum efficiency and unit amplification, for simplicity.) The output field $b_{\text{out}}(t)$ is related to the internal field and the input field by

$$b_{\text{out}}(t) = \sqrt{\kappa} a(t) - b_{\text{in}}(t) \,. \tag{12.20}$$

We will assume the input field to be in the vacuum state. In that case

$$\overline{i(t)} = e\xi \kappa \bar{n} \ , \tag{12.21}$$

where $\bar{n}$ is the mean photon number inside the cavity.

To determine the noise properties of the output field, the appropriate detector quantity is $\overline{i(0)i(\tau)}$. The theory of photo-electron detection (Chap. 3) enables this to be related to the intensity fluctuations by

$$\begin{aligned} \overline{i(0)i(\tau)} = {} & e\xi \langle b_{\text{out}}^{\dagger}(0) b_{\text{out}}(0) \rangle \delta(\tau) + e^2 \xi^2 \langle b_{\text{out}}^{\dagger}(0) b_{\text{out}}(0) \rangle^2 \\ & + e^2 \xi^2 \langle : I(0), I(\tau) : \rangle \end{aligned} \tag{12.22}$$

where : : denotes normal and time ordering and

$$I(\tau) = b_{\text{out}}^{\dagger}(\tau) b_{\text{out}}(\tau). \tag{12.23}$$

The first two terms in (12.22) represent a dc term and a δ-correlated shot-noise term. The last term carries information on a possible reduction in intensity fluctuations. We now define the normalised power spectrum

$$P(\omega) = \frac{2}{e^2 \xi^2} \int_0^{\infty} \mathrm{d}\tau \cos(\omega\tau) \overline{i(0)i(\tau)} \,. \tag{12.24}$$

Using (12.20) one may show that

$$\langle : I(0), I(\tau) : \rangle = \kappa^2 (\langle a^{\dagger}(0) a^{\dagger}(\tau) a(\tau) a(0) \rangle - \langle a^{\dagger}(0) a(0) \rangle^2) \,. \tag{12.25}$$

Thus

$$\frac{\overline{i(0)i(\tau)}}{e^2 \xi^2} = \kappa\bar{n}(1 - \kappa\bar{n}) + \kappa\bar{n}\delta(\tau) + \kappa^2 g(\tau) \tag{12.26}$$

where

$$g(\tau) = \langle a^{\dagger}(0) a^{\dagger}(\tau) a(\tau) a(0) \rangle \,. \tag{12.27}$$

We are only interested in the steady state fluctuations of the output field. In which case we can show that $g(\tau) = \bar{n}^2$ and thus

$$P(\omega) = \kappa \bar{n} \,. \tag{12.28}$$

This flat photocurrent spectrum is the shot-noise limit of the laser.

12.3 Laser Linewidth

Well above threshold the laser produces Poisson photon statistics. A coherent state has the same photon statistics, and this suggests that well above threshold the laser might be producing a coherent state. This is not the case. While the intensity of the laser is stabilised with a Poissonian distribution the phase of the laser undergoes a diffusion process. The effect of this phase diffusion is to cause a decay in the mean amplitude of the laser field, as the phase becomes uniformly distributed over 2π. The rate of amplitude decay Γ is thus a direct measure of the phase diffusion rate.

We will only discuss the case $\gamma_1 = \gamma_2 = \gamma$. The mean amplitude is defined by

$$\langle a(t)\rangle = \sum_{n=0}^{\infty} n^{1/2} \rho_{n,n-1}(t) \,. \tag{12.29}$$

Using (12.10) we find

$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = -\frac{G}{2} \sum_{n=0}^{\infty} \frac{(1/4n_\mathrm{s}) - 1}{1 + (2n+1)/2n_\mathrm{s}} \sqrt{n} \rho_{n,n-1} \,. \tag{12.30}$$

Assuming the laser operates well above threshold we can replace n by $\bar{n}$ in the denominator of each coefficient. Then as $\bar{n} \gg n_\mathrm{s}$

$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = -\frac{G}{8\bar{n}} \langle a\rangle \,. \tag{12.31}$$

Thus the phase diffusion rate is inversely proportional to the intensity of the laser. All second-order phase dependent correlation functions will decay at a similar rate. In particular, the two-time correlation function

$$F(\tau) = \langle a^\dagger(\tau) a(0)\rangle \tag{12.32}$$

will decay at the rate $\Gamma = G/8\bar{n}$, i.e.

$$F(\tau) = \bar{n} \mathrm{e}^{-\Gamma\tau} \,. \tag{12.33}$$

The Fourier transform of this function defines the laser spectrum

$$S(\omega) = \frac{\bar{n}}{\omega^2 + \Gamma^2} \,, \tag{12.34}$$

and thus the laser linewidth is simply

$$\Gamma = \frac{G}{8\bar{n}} . \tag{12.35}$$

It must be emphasised, however, that these results only apply well above threshold.

12.4 Regularly Pumped Laser

The Poissonian photon statistics of a laser reflect the contributions from the random pumping mechanism and spontaneous emission, which lead to an irregular photo-emission sequence. By suppressing the pump fluctuations, sub-Poissonian photon statistics and thus sub-shot-noise photo-current fluctuations, are possible. This has been demonstrated in recent experiments by *Machida* et al. [8] and also *Richardson* and *Shelby* [9] with semiconductor lasers. The pump amplitude fluctuations were reduced by high impedance suppression of the electron injection rate.

We shall demonstrate how the *Scully–Lamb* laser theory can readily be modified to incorporate regular pumping. Regular pumped lasers have been considered theoretically by a number of researchers [10, 11, 12, 13, 14]. We shall follow the approach of *Golubov* and *Sokolov* [10], with some modifications.

Consider a time interval Δt short compared to the time scale on which the field is changing due to damping through the end mirrors. However, the time Δt is very long compared to the time interval between successive pumping atoms entering the cavity. Divide the interval Δt into N steps of length τ. The probability for an excited atom to enter the cavity at time $t_j = j\tau$ is defined to be p. The fundamental probabilities of interest are then the probability of r excited atoms to enter the cavity at any of the N time steps, over the interval Δt. These probabilities are

$$\begin{aligned} P_r(\Delta t) = & \frac{\Delta t(\Delta t - \tau)(\Delta t - 2\tau)\ldots(\Delta t - r\tau)}{\tau^r r!} \left(\frac{p}{1-p}\right)^r \\ & \times \exp\left[\frac{\Delta t}{\tau}\ln(1-p)\right] . \end{aligned} \tag{12.36}$$

To first order in Δt this is

$$P_r(t) = (-1)^{r+1}\left(\frac{p}{1-p}\right)^r \frac{\Delta t}{r\tau} . \tag{12.37}$$

Between each atom entering the cavity the field evolves freely according to

$$\mathscr{T}(\tau) = \mathrm{e}^{-\mathrm{i}\omega_\mathrm{a}\tau a^\dagger a}\rho\,\mathrm{e}^{+\mathrm{i}\omega_\mathrm{a}\tau a^\dagger a} . \tag{12.38}$$

The change in the state of the field due to the passage of a single atom is given by (12.4). It is a simple matter to prove that the operation describing the effect of the pump atoms $\mathscr{P}$ commutes with the free evolution operator $\mathscr{T}$ (if this is not the case a simple master equation for the field state cannot be obtained in general). Thus the change in the state of the field over a time Δt is

$$\rho(t+\Delta t) = \mathscr{T}(\Delta t)\left[\sum_{n=0}^{N} P_n(\Delta t)\mathscr{P}^n\rho(t)\right] \tag{12.39}$$

(i.e., we can factor out the free evolution between each time step). We henceforth assume we are working in the interaction picture and drop the free evolution term. As we assume $\tau \ll \Delta t$ we extend the upper limit on the sum to ∞, then

$$\begin{aligned}\rho(t+\Delta t) &= \rho(t) + \frac{\Delta t}{\tau}\ln(1-p)\rho(t)\\ &\quad + \frac{\Delta t}{\tau}\left[\frac{p}{1-p}\mathscr{P} - \frac{1}{2}\left(\frac{p}{1-p}\right)^2\mathscr{P}^2 + \ldots\right]\rho(t)\,.\end{aligned} \tag{12.40}$$

From which we obtain

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{1}{\tau}\ln(1-p)\rho(t) + \frac{1}{\tau}\ln\left(1+\frac{p}{1-p}\mathscr{P}\right)\rho(t) \tag{12.41}$$

$$= R\ln(1+p\mathscr{U})\rho(t)\,, \tag{12.42}$$

where $R = \tau^{-1}$ is the pumping rate for a perfectly regular process $(p = 1)$ and

$$\mathscr{U} = \mathscr{P} - 1\,. \tag{12.43}$$

We can define an average injection rate $r = pR$, then

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{r}{p}\ln(1+p\mathscr{U})\rho(t)\,. \tag{12.44}$$

In this form we can take the Poisson limit defined by $p \to 0$, $R \to \infty$, such that $pR = \text{constant} = r$. In this limit the equation reduces to that for a normal laser.

The difficulty in discussing the regularly pumped laser is the logarithm term in (12.42). As $\mathscr{U}$ represents the change in the state of the field due to a single atom we might expect $\mathscr{U}$ to be in some sense small. With this in mind we expand the logarithm to second order. Unfortunately this leads to a rather pathological master equation. However, the procedure does give accurate results for the first-and second-order moments of the photon number.

The photon number distribution now obeys the equation

$$\begin{aligned}\frac{\mathrm{d}p_n}{\mathrm{d}t} &= \kappa[-np_n + (n+1)p_{n+1}] + r(-a_{n+1}P_n + a_n p_{n-1})\\ &\quad + \frac{pr}{2}[-a_{n+1}^2 p_n + a_n(a_n + a_{n+1})p_{n-1} - a_n a_{n-1} p_{n-2}]\,,\end{aligned} \tag{12.45}$$

where

$$a_n = \frac{Gn}{r(1+(n/n_s))} . \tag{12.46}$$

To obtain the stationary state variances is a rather more difficult process than for the Poisson pumped case. The mean photon number above threshold is not changed, however the variance is given by

$$V(n) = \bar{n}\left(1 - \frac{p\gamma_1}{2(\gamma_1+\gamma_2)}\right) + n_s . \tag{12.47}$$

We consider some special cases of this result for regular pumping, $p = 1$. Far above threshold $\bar{n} \gg n_s$ so we may neglect the last term in (12.47). When the decay rates are equal, the photon number variance is

$$V(n) = \frac{3\bar{n}}{4}(\gamma_1 = \gamma_2) . \tag{12.48}$$

In this case spontaneous emission from level $|2\rangle$ is contributing to the noise. This effect may be reduced by increasing the decay rate of the lower level with respect to the upper level, $\gamma_1 \gg \gamma_2$. In this case

$$V(n) = \frac{\bar{n}}{2}(\gamma_1 \gg \gamma_2) . \tag{12.49}$$

Thus the width of the photon number distribution inside the cavity is reduced by half.

We now consider intensity fluctuations of the light emerging from the cavity. We may obtain a solution for the normally ordered two-time correlation function $g(\tau)$, from the master equation (12.45) assuming a Gaussian steady state distribution. The result is

$$g(\tau) = \bar{n}^2 + [V(n) - \bar{n}]\mathrm{e}^{-\delta\tau} , \tag{12.50}$$

where

$$\delta = \kappa \frac{\bar{n}/n_s}{1+(\bar{n}/n_s)} . \tag{12.51}$$

Substituting (12.47 and 12.50) into (12.26) the spectrum of the photocurrent fluctuations is given by

$$P(\omega) = \kappa\bar{n}\left(1 + \frac{2\kappa Q\delta}{\omega^2+\delta^2}\right) , \tag{12.52}$$

where

$$Q = \frac{V(n) - \bar{n}}{\bar{n}} \tag{12.53}$$

$$= -\frac{p\gamma_1}{2(\gamma_1+\gamma_2)} + \frac{n_s}{\bar{n}} . \tag{12.54}$$

The Q parameter measures the deviation of the intracavity field from Poisson statistics. In the limit of regular pumping $(p = 1)$ and far above threshold

$$P(\omega) = \kappa\bar{n}\left(1 - \frac{\gamma_1\kappa^2}{(\gamma_1+\gamma_2)(\kappa^2+\omega^2)}\right) . \tag{12.55}$$

Thus at cavity resonance $(\omega = 0)$,

$$\begin{aligned} P(0) &= \kappa\bar{n}(1+2Q) \\ &= \kappa\bar{n}\frac{\gamma_1}{\gamma_1+\gamma_2} . \end{aligned} \tag{12.56}$$

The first term in the first equation represents the shot-noise contribution. A negative value of Q leads to a reduction below the shot-noise limit. If the decay rates are equal $(\gamma_1 = \gamma_2)$, spontaneous emission is not suppressed and

$$P(0) = \frac{\kappa\bar{n}}{2} , \tag{12.57}$$

which represents a 50% reduction below the shot-noise level. However, in the limit $\gamma_1 \gg \gamma_2$, Q approaches – 0.5 far above threshold and

$$P(\omega) = \kappa\bar{n}\left(1 - \frac{\kappa^2}{\kappa^2+\omega^2}\right) . \tag{12.58}$$

Then at cavity resonance, the fluctuation spectrum is zero. This may be compared with the light inside the cavity where the photon number fluctuations were only reduced by one half. This result has the same interpretation as the limit to the intracavity squeezing in a parametric oscillator; there is a destructive interference from the vacuum fluctuations reflected from the cavity mirror and the reduced noise light emerging from the cavity. This results in no fluctuations in the output light on resonance.

In Fig. 12.4 we show the results of the experiment by *Machida* et al. [8] for a regularly pumped semiconductor laser.

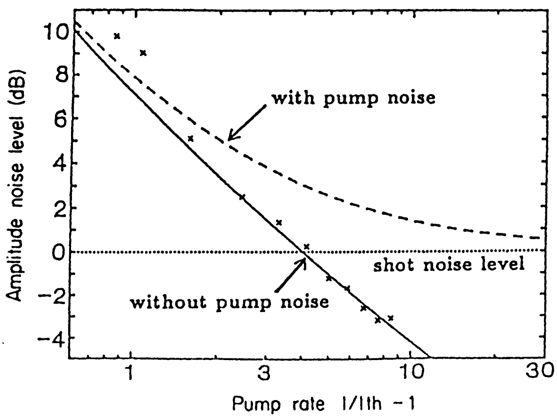

Fig. 12.4 The normalised amplitude noise level versus the pump rate for a laser with pump noise (*dashed*) and with pump noise suppressed (*solid*) [11]

12.A Appendix: Derivation of the Single-Atom Increment

Consider a single multilevel atom (Fig. 12.1) prepared in the state $|2\rangle$. Level $|1\rangle$ is damped at the rate γ_1 to level $|3\rangle$ and level $|2\rangle$ is damped at the rate γ_2 to level $|4\rangle$. Only levels $|1\rangle$ and $|2\rangle$ interact with the cavity field. The master equation describing the dynamics of this system is

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} =& \mathrm{i}g[a^\dagger\sigma_-^{12} + a\sigma_+^{12}, \rho] \\ &+ \frac{\gamma_1}{2}(2\sigma_-^{13}\rho\sigma_+^{13} - \sigma_+^{13}\sigma_-^{13}\rho - \rho\sigma_+^{13}\sigma_-^{13}) \\ &+ \frac{\gamma_2}{2}(2\sigma_-^{24}\rho\sigma_+^{24} - \sigma_+^{24}\sigma_-^{24}\rho - \rho\sigma_+^{24}\sigma_-^{24}) \end{aligned} \tag{12.59}$$

(we ignore spontaneous emission on the lasing levels $|1\rangle, |2\rangle$). We will present a complete operator solution to the master equation over the time τ and then consider the limit $\gamma_i\tau \gg 1$.

Define the operation

$$\mathscr{J}\rho = \gamma_1\sigma_-^{13}\rho\sigma_+^{13} + \gamma_2\sigma_-^{24}\rho\sigma_+^{24} \tag{12.60}$$

and the rate operator

$$R = \gamma_1\sigma_+^{13}\sigma_-^{13} + \gamma_2\sigma_+^{24}\sigma_-^{24} \tag{12.61}$$

$$= \gamma_1|1\rangle\langle 1| + \gamma_2|2\rangle\langle 2| \,. \tag{12.62}$$

The solution to the master equation may then be written formally

$$\begin{aligned}\rho(t) =& \mathscr{S}(t)\rho(0) + \int_0^t \mathrm{d}t_1 \mathscr{S}(t-t_1)\mathscr{J}\mathscr{S}(t_1)\rho(0) \\ &+ \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \mathscr{S}(t-t_1)\mathscr{J}\mathscr{S}(t_1-t_2)\mathscr{J}\mathscr{S}(t_2)\rho(0) \\ &+ \cdots \,, \end{aligned} \tag{12.63}$$

where

$$\mathscr{S}(t)\rho = \mathscr{B}(t)\rho\mathscr{B}^\dagger(t) \,, \tag{12.64}$$

with

$$\mathscr{B}(t) = \exp[-\mathrm{i}g(a^\dagger\sigma_-^{12} + a\sigma_+^{12}) - \gamma_1 t|1\rangle\langle 1| - \gamma_2 t|2\rangle\langle 2|] \,. \tag{12.65}$$

For

$$\rho(0) = |2\rangle\langle 2| \otimes \rho_{\mathrm{F}}(0) \tag{12.66}$$

$$= \sum_{n,\, m=0}^{\infty} \rho_{nm}(0)|n,2\rangle\langle m,2|\,, \tag{12.67}$$

with

$$|n,2\rangle = |n\rangle_{\mathrm{F}} \otimes |2\rangle\,. \tag{12.68}$$

Note that after the action of $\mathscr{J}$ the atom is in a mixture of the states $|3\rangle$ and $|4\rangle$ and is then decoupled from the field. All further action of $\mathscr{S}(t)$ is just the identity, and $\mathscr{J}$ destroys the state. The series thus truncates at first order.

Now one may use the eigenstates of the free Hamiltonian (Chap. 10)

$$|n,\,+\rangle = \frac{1}{\sqrt{2}}(|n,\,2\rangle + |n+1,\,1\rangle)\,, \tag{12.69}$$

$$|n,\,-\rangle = \frac{1}{\sqrt{2}}(|n,\,2\rangle - |n+1,\,1\rangle)\,, \tag{12.70}$$

to show that

$$\begin{aligned}\mathscr{S}(t)(|n,\,2\rangle\langle m,\,2|) =&(c_n^+(t)|n,\,+\rangle + c_n^-(t)|n,\,-\rangle)\\ &\times(\langle m,+|c_m^+(t)^* + \langle m,-|c_m^-(t)^*)\end{aligned} \tag{12.71}$$

where

$$\begin{aligned}c_n^+(t) =&\frac{-\mathrm{i}\exp\left(-\frac{\gamma_+ t}{2}\right)}{2\sqrt{2}\Delta\Omega(n)}\left\{\left[-\mathrm{i}\Omega(n)(1-\Delta) + \frac{\gamma_-}{2}\right]\mathrm{e}^{\mathrm{i}\Delta\Omega(n)t}\right.\\ &\left.+\left[\mathrm{i}\Omega(n)(1+\Delta) - \frac{\gamma_-}{2}\right]\mathrm{e}^{-\mathrm{i}\Delta\Omega(n)t}\right\}\end{aligned} \tag{12.72}$$

and

$$\begin{aligned}c_n^-(t) =&\frac{-\mathrm{i}\exp\left(-\frac{\gamma_+ t}{2}\right)}{2\sqrt{2}\Delta\Omega(n)}\left\{\left[\mathrm{i}\Omega(n)(1+\Delta) + \frac{\gamma_-}{2}\right]\mathrm{e}^{\mathrm{i}\Delta\Omega(n)t}\right.\\ &\left.+\left[-\mathrm{i}\Omega(n)(1-\Delta) - \frac{\gamma_-}{2}\right]\mathrm{e}^{-\mathrm{i}\Delta\Omega(n)t}\right\}\end{aligned} \tag{12.73}$$

where

$$\gamma_\pm = \frac{1}{2}(\gamma_1 \pm \gamma_2)\,, \tag{12.74}$$

$$\Delta = \left(1 - \frac{\gamma_-^2}{4\Omega(n)^2}\right)^{1/2}\,, \tag{12.75}$$

$$\Omega(n) = g\sqrt{n+1}\,. \tag{12.76}$$

We now assume $\gamma_{1,\,2}t \gg 1$. The first term in (12.63) may be ignored as it simply decays to zero.

We are interested in the state of the field alone which is obtained by tracing out over the atomic states. We use

$$\begin{aligned}\mathrm{Tr_A}(\mathscr{J}\mathscr{S}(t)|n,\,2\rangle\langle m,\,2|) = &\frac{\gamma_2}{2}|n\rangle\langle m|[c_n^+(t)+c_n^-(t)][c_m^+(t)^*+c_m^-(t)^*]\\ &+\frac{\gamma_1}{2}|n+1\rangle\langle m+1|[c_n^+(t)-c_n^-(t)]\\ &\times[c_m^+(t)^*-c_m^-(t)^*]\,. \end{aligned} \tag{12.77}$$

Thus we obtain in the steady state, the single atom increment

$$\rho' = \sum_{n,\,m=0}^{\infty}\rho_{nm}(0)(A_{nm}|n\rangle\langle m|+B_{nm}|n+1\rangle\langle m+1|)\,, \tag{12.78}$$

where

$$A_{nm} = \frac{\gamma_2}{2}\int_0^\infty \mathrm{d}t[c_n^+(t)+c_n^-(t)][c_m^+(t)^*+c_m^-(t)^*]\,, \tag{12.79}$$

$$B_{nm} = \frac{\gamma_1}{2}\int_0^\infty \mathrm{d}t[c_n^+(t)-c_n^-(t)][c_m^+(t)^*-c_m^-(t)^*]\,. \tag{12.80}$$

Note that $\mathrm{Tr}(\rho)=1$ requires that $A_{nn}+B_{nn}=1$. We quote only the results for the diagonal matrix elements,

$$A_{nn} = \left(\frac{\gamma_2}{2\gamma_+}\right)\frac{4\Omega(n)^2+2\gamma_1\gamma_+}{4\Omega(n)^2+\gamma_1\gamma_2} \tag{12.81}$$

and

$$B_{nn} = 1-A_{nn} = \left(\frac{\gamma_2}{2\gamma_+}\right)\frac{4\Omega(n)^2}{4\Omega(n)^2+\gamma_1\gamma_2}\,. \tag{12.82}$$

To compute the change in the state we write

$$\rho' = (1+\mathscr{U})\rho = \mathscr{P}\rho\,. \tag{12.83}$$

The diagonal matrix elements of $\mathscr{U}\rho$ are then found to be

$$\langle n|\mathscr{U}\rho|n\rangle = -a_{n+1}p_n + a_n p_{n+1}\;\;, \tag{12.84}$$

where

$$a_{n+1} = A_{nn}-1\,. \tag{12.85}$$

Exercises

12.1 Show that below threshold $(G < \kappa)$ the master equation may be approximated by

$$\frac{d\rho}{dt} = \frac{G}{2}(2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger) + \kappa(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \ .$$

Thus demonstrate that the steady state density operator is

$$\rho^{ss} = \left(1 - \frac{G}{\kappa}\right) \sum_{n=0}^{\infty} \left(\frac{G}{\kappa}\right)^n |n\rangle\langle n| \tag{12.86}$$

which is equivalent to a chaotic state.

12.2 Show that well above threshold the laser master equation may be approximated by

$$\frac{d\rho}{dt} = \frac{Gn_s}{2}(2n^{-1/2}a^\dagger \rho a n^{-1/2} - an^{-1}a^\dagger \rho - \rho an^{-1}a^\dagger) + \frac{\kappa}{2}(2a\rho a^\dagger - a^\dagger \rho a - \rho a^\dagger a)$$

where $n = a^\dagger a$. Show that the steady-state solution is

$$\rho^{ss} = \exp\left(-\frac{Gn_s}{\kappa}\right) \sum_{n=0}^{\infty} \frac{(Gn_s/\kappa)^n}{n!} |n\rangle\langle n| \ . \tag{12.87}$$

12.3 Show that the contours of the Q-function for the laser steady states in Exercises 12.1, 12.2 are: (a) Circles centred on the origin for below threshold, (b) annulli centered at the radius $r = (Gn_s/\kappa)^{1/2}$, for the above threshold state. Thus in both cases the phase of the field is random.

References

1. H. Haken: *Laser Theory*, Reproduction from *Handbuch der Physik* (Springer, Berlin, Heidelberg 1984)
2. M. Sargent III, M.O. Scully, W.E. Lamb: *Laser Physics* (Addison Wesley 1974)
3. M.O. Scully, W.E. Lamb: Phys. Rev. **159**, 208 (1967)
4. M. Lax, W.H. Louisell: Phys. Rev. **185**, 568 (1969)
5. F.T. Arecchi: Phys. Rev. Lett. **15**, 912 (1965)
6. E.A. Johnson, R. Jones, T.P. McLean, E.R. Pike: Phys. Rev. Lett. **16**, 589 (1966)
7. B.L. Morgan, L. Mandel: Phys. Rev. Lett. **16**, 1012 (1966)
8. S. Machida, Y. Yamamoto, Y. Itaya: Phys. Rev. Lett. **58**, 1000 (1987)

9. W. Richardson, R.E. Shelby: Phys. Rev. Lett. **64**, 400 (1990)
10. Y.M. Golubov, I.V. Sokolov: Sov. Phys. JETP **60**, 234 (1984)
11. Y. Yamamoto, S. Machida, O. Nilsson: Phys. Rev. A **34**, 4025 (1986)
12. F. Haake, S.M. Tan, D.F. Walls: Phys. Rev. A **40**, 7121 (1989)
13. J. Bergou, L. Davidovich, M. Orszag, C. Berkert, M. Hillery, M.O. Scully: Opts. Commun. **72**, 82 (1989)
14. M. Marte, P. Zoller: Quantum Optics **2**, 229 (1990)